Mary Robertson.

An action of $\widehat{GT}$ on the stable curves of genus 0.

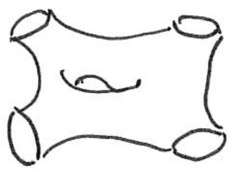
15 November 2017.

Joint w/ Pedro Boavida, Geoffrey Horel.

Uses joint work with Hackney and Yau.

I) Teichmüller towers.

Σ_g^n oriented surface with n boundaries.



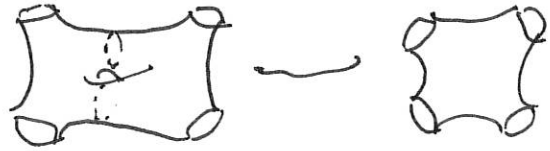
$$\Gamma_g^n = \left\{ \begin{matrix} \text{diffeomorphisms} \\ \text{that fix} \\ \text{boundaries} \end{matrix} \right\} \Big/ \text{isotopy}.$$

Def. An (ideal) Teichmüller tower

(1) $\{\Sigma_g^n\}$ $(n,g) \geq 0$

(2) $\Gamma_{g'}^{n'} \hookrightarrow \Gamma_g^n$ homomorphisms

$\Sigma_{g'}^{n'} \hookrightarrow \Sigma_g^n$ if we can obtain $\Sigma_{g'}^{n'}$ by cutting along closed curves.



Γ_1^4 Γ_0^4

Now, replace this all with profinite completions $\widehat{\Gamma}_g^n = \pi_1^{\text{ét}} \mathcal{M}_g^n$.

Conjecture (Grothendieck).

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \simeq \mathrm{Out}(\{\hat{\Gamma}_g^n\})$$

Outer automorphisms of the tower.

Not proving this today.

Drinfeld introduced an explicitly defined (profinite) group $\widehat{GT}$ with a commutative diagram

$$\begin{tikzcd} & \mathrm{Out}(\{\hat{\Gamma}_g^n\}) \\ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \arrow[ur, "\text{Belyi}"] \arrow[r, hook, "\text{Drinfeld}"'] & \widehat{GT} \arrow[u, "\text{Drinfeld}"'] \end{tikzcd}$$

Thm (Fresse, Willwacher, Turchin, ...).

$$GT^{\mathbb{Q}} \simeq \pi_0 \mathrm{Aut}^h(\mathbb{E}_2^{\mathbb{Q}}).$$

Pro unipotent. So, not related to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Thm (Boavida-~~Horel~~-Robertson). $\widehat{GT} \simeq \pi_0 \mathrm{Aut}^h(\widehat{\Pi}_{0,*+1})$.

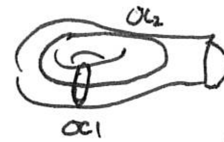
$\hat{\Pi}$ has a non-trivial action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Def (Non-standard). Let $\widehat{GT}^{(1)}$ be the set of all $f \in \hat{F}_2$ (prof. completion of free group on x, y.) which satisfy the property that any

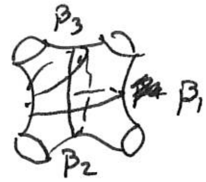
$$\begin{aligned} \hat{F}_2 &\longrightarrow \hat{F}_2 \\ x &\longmapsto x \\ y &\longmapsto f^{-1} x f \end{aligned}$$

extends to an aut. of $\hat{F}_2$ and relations I, II, III.

I) $f(a_2^2, a_1^2)\, f(a_1^2, a_2^2) = 1$ in $\hat{\Gamma}_1^1$.

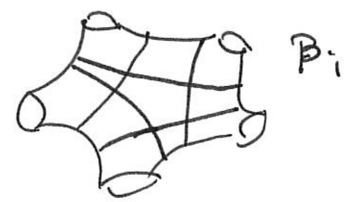


II) $f(b_3, b_1)\, f(b_2, b_3)\, f(b_1, b_2) = 1$ in $\hat{\Gamma}_0^4$. Associator Pentagon



III) $f(b_3, b_4)\, f(b_5, b_1)\, f(b_2, b_3)\, f(b_4, b_5)\, f(b_1, b_2) = 1$ in $\hat{\Gamma}_0^5$.

Hexagon



Def. An ∞-cyclic operad X is

0) $X : \Xi^{op} \longrightarrow sSet$ (∞-groupoids)

1) $X_{(1)} \simeq *$ (single colored)

2) $X_T \xrightarrow{\ \simeq\ } \prod_{v \in T} (X_{c_v})_f$ (Segal condition.)

Same fibrancy thing.

Ξ is a category with objects unrooted trees (connected, simply connected graphs),

E.g. 

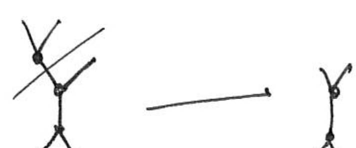
morphisms generated by three classes of maps:

Ξ^- {

-  killing a 1-1 vertex.

-  inner edge contraction, inner coface map,

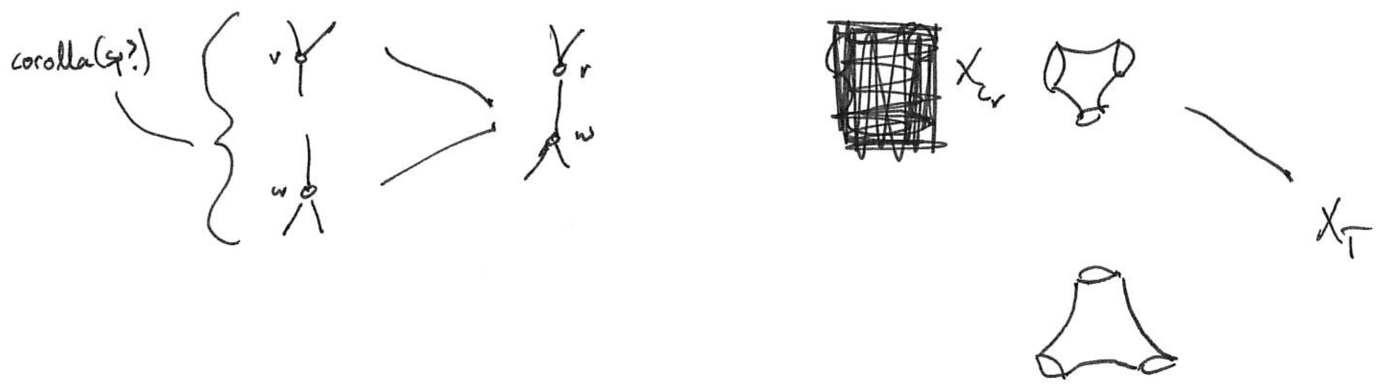
Ξ^+ {

-  pruning outer coface.

Ex. $\overline{M}_{0,*+1} : \Xi^{op} \longrightarrow sSet$.

T =  = $X_T \in$ Nerve(Γ_0^4)

Condition (2)



Recovers Getzler - Kontsevich
cyclic operad of moduli spaces.

$\mathrm{End}(\overline{M}_{0,*+1})$

Determined by
what happens to 
subject to conditions
II and III.

Fact. Let X, Y be two top. spaces.

$$\widehat{X \times Y} \longrightarrow \hat{X} \times \hat{Y}$$

Not typically an $\cong$.

Prop. If X, Y are connected and all homotopy groups of X, Y are good in the sense of Serre, then it is.

Thm (Deligne). $\pi_1 \overline{M}_{0,n}$ all good.

$$H^*(\widehat{G}, M) \xrightarrow{\cong} H^*(G, M)$$

$\forall *$ and all finite G-modules M.

$$\overline{M}_{0,\#+1} : \Xi^{op} \longrightarrow sSet$$

$$\widetilde{\overline{M}}_{0,n}(T) \xrightarrow{\simeq} \prod \widetilde{\overline{M}}_{0,|G|}(G)$$

Thm. $\pi_0 \mathrm{Aut}^h(\widehat{\overline{M}}_{0,\#+1}) \simeq \widehat{GT}$

Cor.

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & fE_2 \arrow[d] \\ * \arrow[r] & \overline{M}_{0,*+1} \end{tikzcd}$$

Drummond-Cole

Profinite complete.

$$\begin{tikzcd} \widehat{S}^1 \arrow[r, dash] \arrow[d, dash] & \widehat{fE}_2 \simeq \widehat{GT} \arrow[d] \\ * \arrow[r] & \widehat{\overline{M}}_{0,*+1} \end{tikzcd}$$

Leads to an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\widetilde{\overline{M}}_{0,*+1}$.